# Critical exponents for higher-representation sources in 3D $SU(3)$ gauge theory from CFT

**Ferdinando Gliozzi**

*Dipartimento di Fisica Teorica, Università di Torino and*
*INFN, sezione di Torino*
*Via P. Giuria, 1, I-10125 Torino, Italy*
*E-mail:* `gliozzi@to.infn.it`

**Silvia Necco**

*IFIC - Instituto de Física Corpuscular, Edificio Institutos de Investigación*
*Apartado de Correos 22085, E-46071 Valencia - Spain*
`necco@ific.uv.es`

ABSTRACT: We establish an exact mapping between the multiplication table of the irreducible representations of SU(3) and the fusion algebra of the two-dimensional conformal field theory in the same universality class of 3D SU(3) gauge theory at the deconfining point. In this way the Svetitsky-Yaffe conjecture on the critical behaviour of Polyakov lines in the fundamental representation naturally extends to whatever representation one considers. As a consequence, the critical exponents of the correlators of these Polyakov lines are determined. Monte Carlo simulations with sources in the symmetric two-index representation, combined with finite-size scaling analysis, compare very favourably with these predictions.

# Contents

## 1. Introduction

To probe the structure of the vacuum of $SU(N)$ gauge theory we dispose of an infinite variety of external sources transforming as arbitrary irreducible representations of the gauge group. One might wonder whether the information extracted in this way is largely redundant, since the force acting on a colour source in a representation $\mathcal{R}$ built up of $j$ copies of the fundamental representation should depend only on its $N-$ality $k_{\mathcal{R}} \equiv j \pmod{N}$, the reason being that all representations with same $k$ (hence transforming in the same manner under the center $\mathbb{Z}_N$) can be converted into each other by the emission of a proper number of soft gluons.

Actually such a property should be regarded as a feature of the IR limit, valid only when the source in question is very far from the other sources. In such a case this is subjected to a confining force only if the $N-$ality is non-vanishing. At intermediate scales lattice studies have shown long ago that this is not the case. Even sources in the adjoint representation (and therefore blind to the center) feel a linear rising potential, with a string tension larger than that of the fundamental representation [1]. Since then most numerical experiments based on large Wilson loops [2–6] yield string tensions which depend on the specific representation $\mathcal{R}$ of the probe source rather than on its $N-$ality. In the IR limit the heavier $\mathcal{R}-$strings are expected to decay into the string with smallest string tension within the same $N$-ality class, called $k-$string. A theoretical description of such a decay as a level-crossing phenomenon can be found in [7]. For a recent discussion on this subject see [8].

A surprisingly similar problem emerges when considering the $SU(N)$ gauge model at the deconfining point. If the transition is second order, one obvious question concerns the critical behaviour of the Polyakov lines in arbitrary representations. Over the years, many

studies have been dedicated to this subject [9 – 14]. The well-verified Svetitsky- Yaffe (SY) conjecture [15] would place the finite-temperature $SU(N)$ gauge theory in the universality class of $\mathbb{Z}_N$ invariant spin model in one dimension less and with short-range interactions. There is a one-to-one correspondence between the irreducible representations of $\mathbb{Z}_N$ and the $N-$ality values of $SU(N)$. Thus, one is tempted to conclude that the non-abelian nature of $SU(N)$ and therefore whatever difference among sources in different representations with the same $N-$ality should be completely lost at criticality: if only the global $\mathbb{Z}_N$ symmetry matters in characterising the universality class, there appears to be no room for independent exponents for Polyakov loops in different representations with the same $N-$ality.

The surprising result is that sources in higher representations, according to various numerical experiments [10, 12, 13], correspond to different magnetisation exponents, one exponent for each representation. Actually a mean field approximation of the effective $SU(2)$ Polyakov-line action at criticality in the $d \rightarrow \infty$ limit shows that the leading amplitudes of higher representations vanish at strong coupling, and the sub-leading exponents become dominant, thus each higher representation source carries its own critical exponent [13].

A somewhat similar conclusion has been reached in $SU(3)$ and $SU(4)$ critical theories in $2+1$ dimensions, starting from a different point of view [7, 17]: the SY conjecture leads to build up a map between the operator product expansion (OPE) of the Polyakov operators in the gauge theory and the corresponding spin operators in the two-dimensional conformal field theory (CFT) describing the associated spin system at criticality. An interesting property of such a correspondence is that one can define a conserved $N-$ality also on the CFT side and that the number of irreducible representations of the conformal algebra (and therefore the number of critical indices) is *larger* than $N$, thus different Polyakov operators with the same $N-$ality can behave differently with an exactly prescribed critical index.

In this paper we build up explicitly the above-mentioned correspondence between the irreducible representations of $SU(3)$ and the primary fields of the CFT describing the critical behaviour of the $(2+1)$ $SU(3)$ gauge theory (see section 2). This leads to some specific predictions on the critical indices to be associated to Polyakov loop operators in higher representations. In section 3 we compare our predictions with a set of Monte Carlo simulations at criticality, using finite size scaling methods, finding a nice and complete agreement.

## 2. The gauge/CFT correspondence

The effective theory describing the interaction of Polyakov lines of any $d + 1$ gauge theory at a finite temperature $T$ can be described, when all the unrelated degrees of freedom are integrated out, as a $d-$dimensional spin model with a global symmetry coinciding with the center of the gauge group. Such an effective theory has only short-range interactions, as Svetisky and Yaffe (SY) observed long time ago [15]. As a consequence, if the deconfinement transition of the gauge theory is second-order, it is in the same universality class of the order-disorder transition of the corresponding spin model.

Therefore all the universal properties of the deconfining transition can be predicted to coincide with the ones of the dimensionally reduced effective model. These include the values of critical indices, the finite-size scaling behaviour, and the correlation functions at criticality.

becomes particularly predictive because one can apply the methods of CFT.

The main ingredient to fully exploit the predictive power of the SY conjecture is a mapping relating the physical observables of the gauge theory to the operators of the reduced model, as first advocated in [16].

The first entry of such a mapping is intrinsically contained in the SY conjecture, namely the correspondence between the order parameter of the deconfining transition of the gauge system, i.e. the Polyakov line in the fundamental representation $f$, and the order parameter $\sigma$ of spontaneous symmetry breaking transition of the spin model:

$$\text{tr}_f(U_{\vec{x}}) \ \sim \ \sigma(\vec{x}) , \tag{2.1}$$

$U_{\vec{x}}$ is the gauge group element associated to the closed path winding once around the periodic imaginary time direction intersecting the spatial plane at the point $\vec{x}$. The above equivalence is only valid in a weak sense, that is, when the left-hand-side of the equation is inserted in a correlation function of the gauge theory and the right-hand side in the corresponding correlator of the spin model.

One obvious question concerns the CFT operators corresponding Polyakov lines in higher representations. On the gauge side these are naturally generated by a proper combination of products of $\text{tr}_f U$, using repeatedly the operator product expansion (OPE)

$$\text{tr}_f(U_{\vec{x}}) \, \text{tr}_f(U_{\vec{y}}) = \sum_{\mathcal{R} \in f \otimes f} C_{\mathcal{R}}(|\vec{x} - \vec{y}|) \, \text{tr}_{\mathcal{R}}(U_{(\vec{x}+\vec{y})/2}) + \cdots \tag{2.2}$$

where $\mathcal{R}$ indicates any irreducible representation belonging in the decomposition of the direct product $f \otimes f$ and the numerical coefficients $C_{\mathcal{R}}(r)$ are suitable functions; they become powers of $r$ at the critical point. The dots represent the contribution of higher dimensional local operators. For our purposes the relevant property of this kind of OPE is that the local operators contributing to the right-hand-side are classified according to the irreducible representations of the gauge group.

On the CFT side we have a similar structure. The order parameter $\sigma$ belongs to an irreducible representation $[\sigma]$ of the Virasoro algebra [18] and the local operators contributing to an OPE are classified according to the decomposition of the direct product of the Virasoro representations of the left-hand-side operators. Such a decomposition is known as fusion algebra [19] and can be written generically as

$$[\lambda_i] \star [\lambda_j] = c_{ij}^k [\lambda_k] \tag{2.3}$$

where the non-negative integers $c_{ij}^k$ are the fusion coefficients.

The consistency of the SY conjecture requires a suitable mapping between the Clebsh-Gordan decomposition of the direct product of irreducible representations of the gauge group and the fusion algebra of the corresponding CFT.

| $\lambda$ name | $\sigma$ spin field | $\psi$ simple current | $\epsilon$ energy | $\mathbb{I}$ identity | $\sigma^+$ spin field | $\psi^+$ simple current |
|---|---|---|---|---|---|---|
| $x_\lambda$ | $\frac{2}{15}$ | $\frac{4}{3}$ | $\frac{4}{5}$ | $0$ | $\frac{2}{15}$ | $\frac{4}{3}$ |
| $(r,s)$ | $(3,3)$ | $(4,3)$ | $(2,1)\oplus(3,1)$ | $(1,1)\oplus(4,1)$ | $(3,3)$ | $(4,3)$ |
| 3-ality | $1$ | $1$ | $0$ | $0$ | $-1$ | $-1$ |
| $\mathcal{D}_\lambda$ | $\frac{1+\sqrt{5}}{2}$ | $1$ | $\frac{1+\sqrt{5}}{2}$ | $1$ | $\frac{1+\sqrt{5}}{2}$ | $1$ |

**Table 1:** Operator content of CFT describing the critical three-state Potts model.

| | $\sigma$ | $\psi$ | $\epsilon$ | $\mathbb{I}$ | $\sigma^+$ | $\psi^+$ |
|---|---|---|---|---|---|---|
| $\sigma$ | $\sigma^+ + \psi^+$ | $\sigma^+$ | $\sigma + \psi$ | $\sigma$ | $\mathbb{I} + \epsilon$ | $\epsilon$ |
| $\psi$ | $\sigma^+$ | $\psi^+$ | $\sigma$ | $\psi$ | $\epsilon$ | $\mathbb{I}$ |
| $\epsilon$ | $\sigma + \psi$ | $\sigma$ | $\mathbb{I} + \epsilon$ | $\epsilon$ | $\sigma^+ + \psi^+$ | $\sigma^+$ |
| $\mathbb{I}$ | $\sigma$ | $\psi$ | $\epsilon$ | $\mathbb{I}$ | $\sigma^+$ | $\psi^+$ |
| $\sigma^+$ | $\mathbb{I} + \epsilon$ | $\epsilon$ | $\sigma^+ + \psi^+$ | $\sigma^+$ | $\sigma + \psi$ | $\sigma$ |
| $\psi^+$ | $\epsilon$ | $\mathbb{I}$ | $\sigma^+$ | $\psi^+$ | $\sigma$ | $\psi$ |

**Table 2:** Fusion algebra.

## 2.1 The $SU(3)$/3-state Potts-model correspondence

In the present paper we are interested in the 2+1 dimensional $SU(3)$ gauge model which is described at the deconfinement point by the the same universality class of three-state Potts model, as it has been checked in numerical simulations [12].

At the critical point such a model is described by a minimal CFT with central charge $c = \frac{4}{5}$. The local operator content is composed by six primary fields $\lambda$ associated to six irreducible representations $[\lambda]$ that we list in table 1 along with their scaling dimensions $x_\lambda$ and the Kac labels $(r,s)$ of the corresponding Virasoro representations.[1]

These are related to the scaling dimensions by

$$x_{r,s} = \frac{(6r - 5s)^2 - 1}{60} \ . \tag{2.4}$$

The reason why we call $\psi$ a simple current will be clear in the next subsection. The last two rows of table 1 refer to two important properties of the fusion algebra, which is presented in table 2.

First, these fusion rules show that the six representations may be cast into three doublets corresponding to the three values $(1, 0, -1)$ of a multiplicatively conserved quantum

---

[1]Actually the critical three-state Potts model is invariant under a larger algebra than that of Virasoro, the so-called $\mathcal{W}_3$ algebra, and the representations listed in table 1 are irreducible representations of such a larger algebra [20]. In fact the identity and the energy are the sum of two different irreducible Virasoro representations, as table 1 shows. For a recent discussion on the critical three-state Potts model see [21].

number, exactly like the triality in the SU(3) gauge theory. This provides a non-trivial check of SY conjecture.

Another useful notion is that of *quantum dimension* $\mathcal{D}_\lambda$ associated to each representation $\lambda$ of CFT and related in a profound way to the modular properties of the theory [19]. For our purposes we simply define the $\mathcal{D}_\lambda$'s as the set of (real or complex) numbers obeying the sum rule

$$\mathcal{D}_\lambda \, \mathcal{D}_\mu = \sum_{\nu \in \lambda \star \mu} \mathcal{D}_\nu \,, \tag{2.5}$$

which is a property that the quantum dimension shares with the dimension $d_\mathcal{R} = \operatorname{tr}_\mathcal{R} \mathbf{1}$ of a group representation $\mathcal{R}$: this is an integer satisfying the obvious relation $d_\mathcal{R} \, d_\mathcal{S} = \sum_{\mathcal{T} \in \mathcal{R} \otimes \mathcal{S}} d_\mathcal{T}$.

Applying such a definition to the fusion algebra of table 2 one finds at once that the set of representations split into two parts; three of them have $\mathcal{D} = 1$, while the quantum dimensions of the other three is a solution of the quadratic equation $x^2 = x + 1$, as listed in the last row of table 1. The importance of such a notion stems from the fact that the sum rule (2.5) as well as conservation of triality are sufficient conditions to fix uniquely the fusion algebra, as one can check at once.

We are now in a position to add other entries in the gauge/CFT mapping. It suffices to compare the fusion rules of table 2 with the Clebsh-Gordan decomposition of the direct product of the corresponding SU(3) irreducible representations or multiplets. These are labelled by a pair of integers $(p, q)$ which give the numbers of covariant and contravariant indices. The corresponding dimensions $d_{(p,q)}$ and triality $k_{(p,q)}$ are given by

$$d_{(p,q)} = (p + 1)\,(q + 1)\,(p + q + 2)/2 \,, \quad k_{(p,q)} \equiv (p - q)\,\mathrm{mod}\,3 \,. \tag{2.6}$$

To adhere to the standard notation we denote the SU(3) multiplets through their dimension $(p, q) \equiv \{d_{(p,q)}\}$.

Comparison of the fusion rule

$$[\sigma] \star [\sigma^+] = [\mathbb{I}] + [\epsilon] \tag{2.7}$$

with the analogous one on the gauge side

$$\{3\} \otimes \{\bar{3}\} = \{1\} + \{8\} \,, \quad \text{or} \quad f \otimes \bar{f} = 1 + adj \tag{2.8}$$

yields the new entry

$$\operatorname{tr}_{\mathrm{adj}}(U_{\vec{x}}) \ \sim \ a + \epsilon(\vec{x}) \,, \tag{2.9}$$

that is expected to be valid for any SU($N$) gauge theory undergoing a continuous phase transition. The constant $a$ can be numerically evaluated [11] using the expected finite-size behaviour

$$\langle \operatorname{tr}_{\mathrm{adj}}(U) \rangle \simeq a + \frac{b}{L^{2 - 1/\nu}} \,, \tag{2.10}$$

where $L$ is the spatial size of the system. We used the general relation $x_\epsilon = d - 1/\nu$ relating the scaling dimension of the energy operator to the thermal exponent $\nu$.

Similarly the fusion rule

$$[\sigma] \star [\sigma] = [\sigma^+] + [\psi^+] \,, \tag{2.11}$$

corresponds to

$$\{3\} \otimes \{3\} = \{\bar{3}\} + \{6\} \,, \tag{2.12}$$

where the anti-fundamental representation $\{\bar{3}\}$ corresponds to the antisymmetric combination of two quarks while $\{6\}$ is the symmetric one. Owing to the correspondence $\mathrm{tr}_{\bar{f}}(U_{\vec{x}}) \sim \sigma^+(\vec{x})$, the symmetric representation yields the new entry

$$\mathrm{tr}_{\{6\}}(U_{\vec{x}}) \;\sim\; \psi^+(\vec{x}) + c\,\sigma^+(\vec{x}) \,. \tag{2.13}$$

The coefficient $c$ is necessarily different from zero: if it were vanishing, the fusion rule $[\psi^+] \star [\sigma] = [\epsilon]$ would correspond to $\{6\} \otimes \{3\} = \{8\} + \{10\}$ which would in turn imply either $a = 0$ in eq. (2.9), which is not the case, or a dubious cancellation between the contributions of rep.s $\{8\}$ and $\{10\}$. As a consequence, the Polyakov-Polyakov critical correlator of the symmetric representation $\{6\}$ is expected to have the following general form in the thermodynamic limit

$$\langle \mathrm{tr}_{\{6\}}(U_{\vec{x}}) \, \mathrm{tr}_{\{\bar{6}\}}(U_{\vec{y}}) \rangle = \frac{c_s}{r^{2x_\sigma}} + \frac{c_u}{r^{2x_\psi}} \,, \tag{2.14}$$

with $r = |\vec{x} - \vec{y}|$ and $c_s, c_u$ suitable coefficients. Since $x_\sigma < x_\psi$, the second term drops off more rapidly than the first, thus at large distance this correlator behaves like that of the anti-symmetric representation $\{\bar{3}\}$ as expected also at zero temperature.

## 2.2 A difficult question

The SU(3)/CFT correspondence we have just established works in the sense that higher dimensional representations that can be screened to the fundamental are associated with operators that are suppressed in large distance correlators, like the $\{6\}$ decaying in $\{\bar{3}\}$ described by eq. (2.14). A similar property emerges also in SU(4) gauge theory [17].

One might ask the following question: how could the two-dimensional CFT "know" about the requirement of such a hierarchy of scaling dimensions due to the finite-T relation to non-abelian gauge theories? In other words, could one predict an ordering of scaling dimensions in 2D which agrees with the expectations of the corresponding gauge theory?[2]

Since the core of the gauge/CFT correspondence resides in the mapping between the Clebsh-Gordan decomposition of the direct product of representations of the gauge group and the CFT fusion algebra (2.3), the suggestion naturally arises that the ordering of the scaling dimensions could be a direct consequence of the algebraic structure of the latter.

Even if we failed in finding a complete proof of such a hierarchy of scaling dimensions, we still believe that this property should be somehow encoded in the fusion rules. Hints can be found in the relationship between fusion algebra and modular invariance we alluded in the Introduction, which yields restrictions on the allowed values of the scaling dimensions [19]. For instance, in a CFT with a finite number $M$ of primary fields, like in the

---

[2]Actually this is a question posed to one of us by P. Damgaard.

case at hand, one gets [22]

$$3\sum_{i=1}^{M} x_i = c\,\frac{M}{4}\mathrm{mod}\,\mathbb{Z}\,, \tag{2.15}$$

where $c$ is the central charge. Further relevant information comes from the *simple currents*[23]. These are by definition those primary fields $\Psi$ which have unique fusion rules with all primaries of the theory, i.e.

$$\Psi \star \lambda_i = \lambda_j \tag{2.16}$$

with a single primary field $\lambda_j$ appearing on the right-hand side for any choice of $\lambda_i$. Iterating the fusion rule of a simple current with itself generates a group $\mathbb{Z}_N$, with $\Psi^N = \mathbb{I}$. As an example, the primary $\psi$ of table 2 is a simple current with $N = 3$. The scaling dimensions of simple currents are strongly restricted. Monodromy properties of the associated correlators easily yield

$$x_\Psi = m\,\frac{N-1}{N}\,\mathrm{mod}\,\mathbb{Z}\,, \tag{2.17}$$

with some integer $m$ modulo $N$. This condition restricts the scaling dimensions of $\psi$ to two possible values $x_\psi = \frac{2}{3}$ or $x_\psi = \frac{4}{3}$ (which is the correct value). Inserting this constraint into eq. (2.15) we get the further restriction $x_\epsilon + 2\,x_\sigma = \frac{1}{5} + \frac{n}{3}$, with $n$ arbitrary integer, which does not suffice to find the complete solution.

## 3. Monte Carlo simulations

We performed Monte Carlo simulations on the finite-T (2+1)-dimensional SU(3) gauge model with standard plaquette action at the deconfinement point, using two critical couplings estimated in [24], namely $\beta_c = 8.155(15)$ for the temporal extension $N_t = 2$ and $\beta_c = 14.74(5)$ at $N_t = 4$ while the spatial size $L$ of the lattice was chosen in the range $8 \le L \le 64$.

We adopted an hybrid updating algorithm consistent of one heat-bath step combined with $N_{\mathrm{or}}$ over-relaxation steps; in particular we used $N_{\mathrm{or}} = 10$ in all our simulations.[3]

It is well known that in the confined phase expectation values of large Wilson loops and Polyakov loops correlation functions at large distances are difficult to measure, since the signal-to-noise ratio decreases exponentially. For this reason, efficient variance reductions methods have been developed [25]. At the deconfining point the situation is much more favourable because the exponential decay is replaced by a power law; correlation functions at large distances can be measured with good precision without employing special techniques.

For the different extensions $L$ at $N_t = 2$ we collected between 10000 and 30000 measurements. For $N_t = 4$ we collected 10000 measurements for each $L$. Between two measurements we performed 100-200 updating steps. At the critical point update algorithms suffer from critical slowing down and one expects large autocorrelation times of the observables; we analysed our data through jackknife binning by using sufficiently large sizes of the bins in order to take this into account.

---

[3]Notice that we did not perform a systematic study in order to optimise the choice of $N_{\mathrm{or}}$.

### 3.1 Finite-size scaling analysis

Observing the second term of eq. (2.14) is very challenging from a computational point of view, because it drops off much more rapidly than the fist term, being $x_\psi$ rather large (actually $x_\psi = 10\,x_\sigma$). In order to gain a better control of this behaviour one has to resort to finite size scaling analysis.

The critical two-point function of the spin field $\sigma$ in a torus $L, L'$, i.e. in a rectangle with periodic boundary conditions of periods $L$ and $L'$ in the $x$ and $y$ directions respectively, has the finite-size scaling form

$$\langle \sigma(0,0)\sigma^+(x,y) \rangle = L^{-2x_\sigma} f(x/L, y/L') \,, \tag{3.1}$$

where $f$ is a universal function. Unfortunately the general Ansatz for CFT correlation functions on a torus proposed long ago [26] cannot be applied to the present case, therefore the form of $f$ for the 3-state Potts model is substantially unknown, apart from the limit $L' \to \infty$ (cylindrical geometry), where the principle of conformal invariance at the critical point allows to write the explicit, exact form of $f$ in any CFT [27].

In our numerical simulations we put instead $L = L'$. Note that for $r = \sqrt{x^2 + y^2} \ll L$ $f$ becomes a function of the single variable $\frac{r}{L}$. In the thermodynamic limit $L \to \infty$ scaling considerations yield

$$f(x/L, y/L) \to \left( \frac{L}{r} \right)^{2x_\sigma} \,. \tag{3.2}$$

The SY conjecture predicts that, combining (2.1)and (3.1), the Polyakov-Polyakov correlator in fundamental representation, defined as

$$\mathcal{G}_{\{3\}}(x,y) \equiv \langle \mathrm{tr}\,_{\{3\}} U_{(0,0)} \; \mathrm{tr}\,_{\{3\}} U^\dagger_{(x,y)} \rangle \tag{3.3}$$

should behave as

$$\mathcal{G}_{\{3\}}(x,y) \simeq L^{-2x_\sigma} f(\xi_1, \xi_2)\,, \quad \xi_1 = x/L\,, \; \xi_2 = y/L \,. \tag{3.4}$$

It should be stressed that, in writing eq. (3.4), as well as the analogous ones (3.8) and (3.11) below, as with all lattice identifications of scaling operators, the correlators of either side are asymptotically equal only when $L$ is large and the points are far apart. At smaller separations there are additional, less relevant operators on the right-hand side which will give rise to corrections. These are visible in figure 1 where we plotted the quantity $\langle \mathrm{tr}\,_{\{3\}} U_0 \, \mathrm{tr}\,_{\{3\}} U^\dagger_{\frac{r}{L}} \rangle (\frac{L}{L_o})^{2x_\sigma}$ versus $r/L$. The points fall with good accuracy on a single curve - thus verifying the Ansatz (3.4)- only for $r/L > 0.15$. The first correction to scaling of (3.4) is a term proportional to $L^{-2x_\sigma - 2}$:

$$\mathcal{G}_{\{3\}}(x,y) \simeq L^{-2x_\sigma} f(\xi_1, \xi_2) + L^{-2x_\sigma - 2} \bar{f}(\xi_1, \xi_2)\,, \tag{3.5}$$

where $\bar{f}$ is another scaling function. Other corrections come from the fact that the lattice system is not exactly at the critical point, however there is no sign of this kind of corrections within the accuracy of our data.
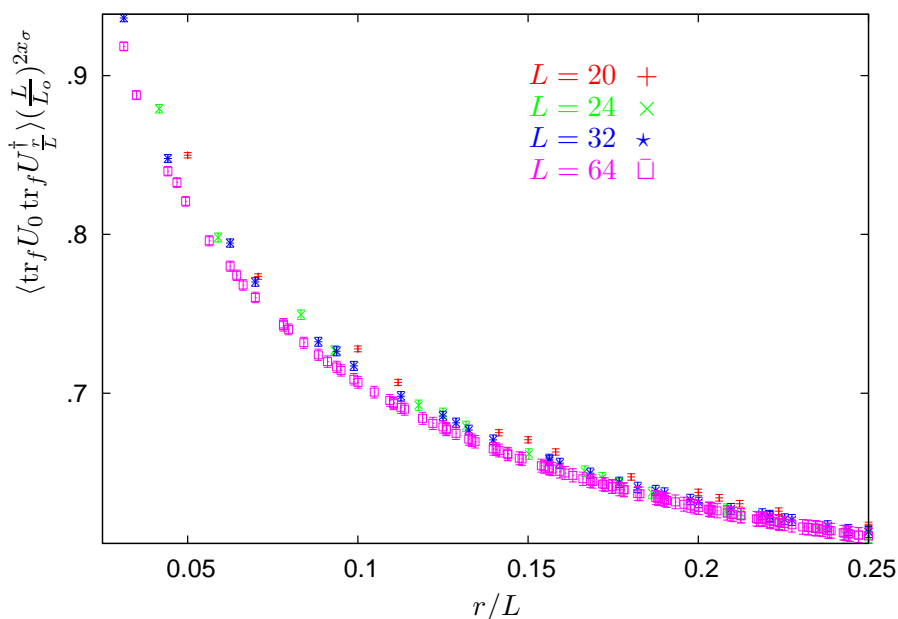
**Figure 1:** Test of the finite-size scaling form (3.5) on a $L \times L \times 2$ lattice. $L_o$ is a reference scale.
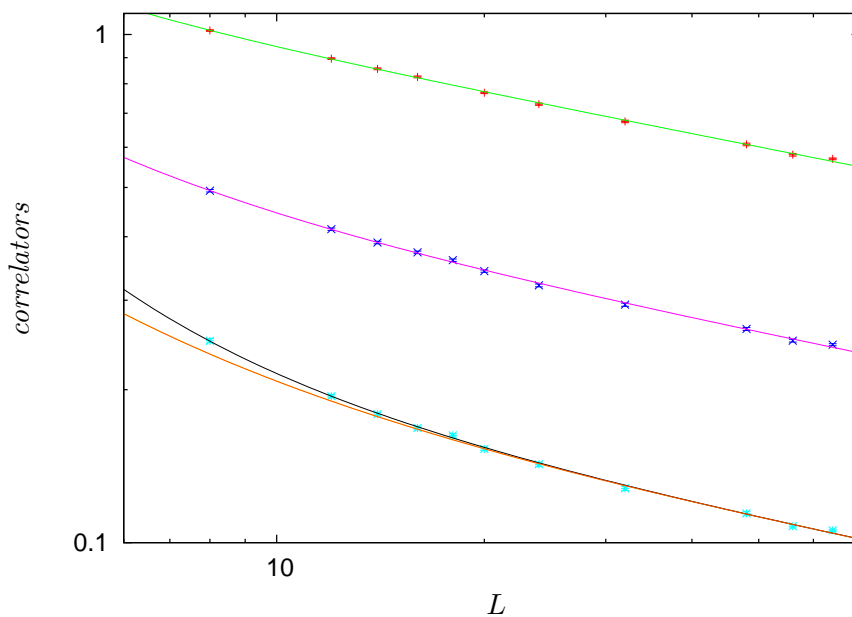


**Figure 2:** Finite-size scaling analysis of the Polyakov-Polyakov correlators at a distance $r = L/2$ in a $L \times L \times 2$ lattice as a function of the size $L$. The top line is a fit of $\langle \mathrm{tr}_{\{3\}} U_{(0,0)} \, \mathrm{tr}_{\{3\}} U^{\dagger}_{(0,\frac{L}{2})} \rangle$ data to (3.5); it is apparently a straight-line in this log log plot, indicating that the corrections to scaling are small; the medium curve is a fit of $\langle \mathrm{tr}_{\{3\}} U_{(0,0)} \, \mathrm{tr}_{\{6\}} U_{(0,\frac{L}{2})} \rangle$ to (3.8). The bottom line is not a fit: it is determined by the other two sets of data by putting $h = 0$ in (3.11). The short distance data show the non-vanishing contribution of the $\langle \psi \, \psi^{+} \rangle$ correlator.

| $i$ | $j$ | $f(i/4, j/4)/f(1/4, 1/4)$ | $g(i/4, j/4)/g(1/4, 1/4)$ | $c$ |
|---|---|---|---|---|
| 1 | 0 | $1.8403(41)/f(1/4, 1/4)$ | $1.556(86)/g(1/4, 1/4)$ | 0.4273(15) |
| 1 | 1 | 0.9616(44) | 0.91(10) | 0.4257(15) |
| 2 | 0 | 0.9485(44) | 0.89(10) | 0.4255(17) |
| 2 | 1 | 0.9369(45) | 0.83(10) | 0.4255(17) |
| 2 | 2 | 0.9268(44) | 0.79(10) | 0.4260(18) |

**Table 3:** Estimates of the universal functions $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ and of the constant $c$ in critical SU(3) lattice models with temporal size $N_t = 2$.

| $i$ | $j$ | $f(i/4, j/4)/f(1/4, 1/4)$ | $g(i/4, j/4)/g(1/4, 1/4)$ | $c$ |
|---|---|---|---|---|
| 1 | 0 | $1.5596(70)/f(1/4, 1/4)$ | $2.50(30)/g(1/4, 1/4)$ | 0.3113(30) |
| 1 | 1 | 0.9597(85) | 0.87(19) | 0.3103(25) |
| 2 | 0 | 0.9468(81) | 0.84(19) | 0.3097(24) |
| 2 | 1 | 0.9360(77) | 0.79(17) | 0.3102(23) |
| 2 | 2 | 0.9264(78) | 0.77(18) | 0.3096(25) |

**Table 4:** Estimates of the universal functions $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ and of the constant $c$ in critical SU(3) lattice models with temporal size $N_t = 4$.

As a next step we considered the mixed correlator $\langle \sigma(0,0)\, \psi^+(x, y) \rangle$. This is zero in the thermodynamic limit because the fusion rule $[\sigma] \star [\psi^+] = [\epsilon]$ does not contain the identity $[\mathbb{I}]$. On a torus it can be expressed in terms of another universal function

$$\langle \sigma(0,0)\psi^+(x,y) \rangle = L^{-x_\sigma - x_\psi} g(\xi_1, \xi_2) \ . \tag{3.6}$$

Again the scaling dimensions of the involved operators imply, for large $L$,

$$g(x/L, y/L) \to \left( \frac{L}{r} \right)^{x_\sigma + x_\psi - x_\epsilon} \ . \tag{3.7}$$

Therefore the new entry (2.13) of the SY conjecture yields

$$\langle \mathrm{tr}_{\{3\}} U_{(0,0)} \, \mathrm{tr}_{\{6\}} U_{(x,y)} \rangle \simeq c\, \mathcal{G}_{\{3\}}(x, y) + L^{-x_\sigma - x_\psi} g(\xi_1, \xi_2) \ . \tag{3.8}$$

The finite-size scaling form of the simple current correlator $\langle \psi(0)\, \psi^+(\vec{x}) \rangle$ is associated to a third universal function $h(x/L, y/L)$

$$\langle \psi(0,0)\psi^+(x,y) \rangle = L^{-2x_\psi} h(\xi_1, \xi_2) \,, \tag{3.9}$$

with

$$h(x/L, y/L) \to \left( \frac{L}{r} \right)^{2x_\psi} \tag{3.10}$$

in the large volume limit. Hence, on the gauge side, the Polyakov correlator in the {6} representation can be written as

$$\left\langle \mathrm{tr}_{\{6\}} U_{(0,0)} \mathrm{tr}_{\{6\}} U^{\dagger}_{(x,y)} \right\rangle \simeq c^2 \, \mathcal{G}_{\{3\}}(x,y) + 2c\frac{g(\xi_1,\xi_2)}{L^{x_\sigma+x_\psi}} + \frac{h(\xi_1,\xi_2)}{L^{2x_\psi}} \; . \tag{3.11}$$

In order to test these finite-size scaling relations it is worth noting that a rescaling of both the lattice size and the distance $r$ by a common factor $s$ is compensated, at criticality, by a rescaling of the correlation function which depends on the scaling dimension of the involved operators, as eq.s (3.4), (3.8) and (3.11) clearly show.

In practice we proceeded as follows. We chose both $x$ and $y$ of the form $jL/4$ with $j = 0, 1, 2$.[4] Varying the linear size $L$ over a set of different values,[5] we generated, for each choice of $r/L$, a sample of data. Since different lattice sizes imply different numerical experiments, our data are by construction statistically independent. A typical set of data is shown in figure 2. In all the cases we do not attempted to use the scaling dimensions as fitting parameters. The $\chi^2$ values of such a kind of fits provide a check on the systematic errors arising from corrections to scaling. In all the fits the $\chi^2/d.o.f$ was in the range between 1 and 2.

Estimates of the universal functions $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ in the five special points considered, as well as the constant $c$ are reported in table 3 ($N_t = 2$) and in table 4 ($N_t = 4$). We consider the fact that these functions on lattices with different temporal extensions are substantially the same within the errors, apart from a different multiplicative normalisation, as a highly non-trivial consistency test of the present description. It is also remarkable the $c$ evaluated on different points $(\xi_1, \xi_2)$ has a constant and stable value, as (3.8) and (3.11) require.

## 3.2 Multiplets of vanishing triality

In a SU(N) gauge theory at finite temperature one could choose as the exact order parameter of the deconfinement transition the Polyakov line in any representation of non-zero $N$-ality. At the deconfining point we have

$$\lim_{L\to\infty} \langle \mathrm{tr}_{\mathcal{R}} U \rangle_{T=T_c} = 0 \, , \qquad \forall \, k_{\mathcal{R}} \neq 0 \; . \tag{3.12}$$

One obvious question concerns the critical behaviour of Polyakov lines corresponding to sources in representations of zero $N$-ality.

A straightforward consequence our gauge/CFT correspondence is that the finite-size behaviour (2.9) of the adjoint representation can be enlarged to any multiplet of vanishing $N$-ality. In the present SU(3) case we can use our Monte Carlo data at $r = 0$ to extract the vacuum expectation value for the 0-triality multiplets of dimensions 8,10 and 27. Indeed, combining (2.8) with the Clebsh-Gordan decompositions

$$\{3\} \otimes \{6\} = \{8\} + \{10\}; \qquad \{\bar{6}\} \otimes \{6\} = \{1\} + \{8\} + \{27\}, \tag{3.13}$$

---

[4]In this way the minimal non-vanishing distance considered ($r = L/4$) lies in the scaling region inferred from figure 1.

[5]Actually the number of different lattice sizes was 10 for $N_t = 2$ and 6 for $N_t = 4$.

| $\mathcal{R}_o$ | $N_t$ | = | 2 | $N_t$ | = | 4 |
|---|---|---|---|---|---|---|
| | $a_{\mathcal{R}_o}$ | | $b_{\mathcal{R}_o}$ | $a_{\mathcal{R}_o}$ | | $b_{\mathcal{R}_o}$ |
| $\{8\}$ | 0.5640(10) | | 0.985(10) | 0.3589(13) | | 1.218(26) |
| $\{10\}$ | 0.0804(20) | | 0.221(25) | 0.3144(26) | | 0.160(52) |
| $\{27\}$ | 0.0592(21) | | 0.181(27) | 0.0167(24) | | 0.101(49) |

**Table 5:** Fits to eq. (3.14) for various representations of zero triality.

and using the new entry (2.13), we get

$$\langle \mathrm{tr}_{\mathcal{R}_o} U \rangle = a_{\mathcal{R}_o} + b_{\mathcal{R}_o} L^{-x_\epsilon} + \cdots \quad \mathcal{R}_o = \{8\}, \{10\}, \{27\}, \tag{3.14}$$

where the ellipses indicate the corrections to scaling.[6] In our case they can be neglected, being the $\chi^2/d.o.f$ values of the fits to (3.14) always less than 1. The estimates of the parameters $a_{\mathcal{R}_o}$ and $b_{\mathcal{R}_o}$ are reported in table 5.

Note that the quantities $a_{\mathcal{R}_o}$ represent the thermodynamic limit of the vacuum expectation value of the Polyakov line in the representation $\mathcal{R}_o$. It turns out that representations of zero $N$-ality yield $a_{\mathcal{R}_o} \neq 0$. Similar results have also been found in 4D gauge theories near the deconfining point [13, 14].

## 4. Conclusion

In this paper we presented a consistent extension of the Svetitsky-Yaffe conjecture in (2+1) dimensional SU(3) gauge theory at finite temperature which led us to associate Polyakov lines in arbitrary representations of the gauge group to suitable conformal operators of the corresponding 2D CFT.

In particular, we built up a correspondence between the multiplication table of the irreducible representations of the gauge group and the fusion algebra of the primary operators of the critical 3-state Potts model.

One important consequence is that the critical exponents of the correlators of these Polyakov loops are univocally determined. We studied in particular the Polyakov line in the symmetric, two-index representation $\{6\}$ and determined the functional finite-size form of some related correlator. We also discussed the vacuum expectation value of a single Polyakov line in the first few multiplets of vanishing triality. We tested these predictions in high precision Monte Carlo simulations finding complete agreement.

### Acknowledgments

---

[6]The first two corrections are the first descendent field of $[\mathbb{1}]$ which gives a contribution $\propto L^{-2}$ and the subleading thermal operator $\epsilon'$, corresponding to the Kac labels $(3,1)$ of table 1, which gives a contribution $\propto L^{-x_{\epsilon'}}$ with $x_{\epsilon'} = \frac{14}{5}$.

## References

[1] C.W. Bernard, *Adjoint Wilson lines and the effective gluon mass*, *Nucl. Phys.* **B 219** (1983) 341;
J. Ambjørn, P. Olesen and C. Peterson, *Stochastic confinement and dimensional reduction, 2. Three-dimensional* SU(2) *lattice gauge theory*, *Nucl. Phys.* **B 240** (1984) 533.

[2] N.A. Campbell, I.H. Jorysz and C. Michael, *The adjoint source potential in* SU(3) *lattice gauge theory*, *Phys. Lett.* **B 167** (1986) 91.

[3] G.I. Poulis and H.D. Trottier, *'Gluelump' spectrum and adjoint source potential in lattice QCD in three-dimensions*, *Phys. Lett.* **B 400** (1997) 358 [`hep-lat/9504015`];
C. Michael, *Adiabatic surfaces from the lattice: excited gluonic potentials*, `hep-ph/9809211`.

[4] S. Deldar, *Static* SU(3) *potentials for sources in various representations*, *Phys. Rev.* **D 62** (2000) 034509 [`hep-lat/9911008`].

[5] G.S. Bali, *Casimir scaling of* SU(3) *static potentials*, *Phys. Rev.* **D 62** (2000) 114503 [`hep-lat/0006022`].

[6] C. Piccioni, *Casimir scaling in* SU(2) *lattice gauge theory*, *Phys. Rev.* **D 73** (2006) 114509 [`hep-lat/0503021`].

[7] F. Gliozzi, *The decay of unstable k-strings in* SU($N$) *gauge theories at zero and finite temperature*, *JHEP* **08** (2005) 063 [`hep-th/0507016`].

[8] M. Shifman, *k strings from various perspectives: QCD, lattices, string theory and toy models*, *Acta Phys. Polon.* **B36** (2005) 3805 [`hep-ph/0510098`].

[9] M. Gross and J.F. Wheater, *Scales of deconfinement?*, *Phys. Rev. Lett.* **54** (1985) 389;
P.H. Damgaard, *The adjoint string at finite temperature*, *Phys. Lett.* **B 183** (1987) 81;
H. Markum, M. Faber and M. Meinhart, SU(3) *multiplet sources at the phase transition to the quark gluon plasma*, *Phys. Rev.* **D 36** (1987) 632.

[10] P.H. Damgaard, *The free energy of higher representation sources in lattice gauge theories*, *Phys. Lett.* **B 194** (1987) 107;
K. Redlich and H. Satz, *Higher representation Wilson lines in* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 213** (1988) 191;
J. Christensen and P.H. Damgaard, *Critical behavior at the deconfinement phase transition of* SU(2) *lattice gauge theory in (2+1)-dimensions*, *Nucl. Phys.* **B 348** (1991) 226;
*Finite size scaling and conformal symmetry around the (2+1)-dimensional* SU(2) *deconfinement phase transition*, *Nucl. Phys.* **B 354** (1991) 339.

[11] J.E. Kiskis, *Behavior of higher representation Wilson lines in finite temperature,* SU(2)*, lattice gauge theory*, *Phys. Rev.* **D 41** (1990) 3204;
J.E. Kiskis and P.M. Vranas, *Adjoint wilson line in* SU(2) *lattice gauge theory*, *Phys. Rev.* **D 49** (1994) 528 [`hep-lat/9307012`].

[12] J. Christensen, G. Thorleifsson, P.H. Damgaard and J.F. Wheater, *Thermodynamics of* SU(3) *lattice gauge theory in (2+1)-dimensions*, *Nucl. Phys.* **B 374** (1992) 225.

[13] P.H. Damgaard and M. Hasenbusch, *Screening and deconfinement of sources in finite temperature* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 331** (1994) 400 [`hep-lat/9404008`].

[14] A. Dumitru, Y. Hatta, J. Lenaghan, K. Orginos and R.D. Pisarski, *Deconfining phase transition as a matrix model of renormalized Polyakov loops*, *Phys. Rev.* **D 70** (2004) 034511 [`hep-th/0311223`].

[15] B. Svetitsky and L.G. Yaffe, *Critical behavior at finite temperature confinement transitions*, *Nucl. Phys.* **B 210** (1982) 423.

[16] F. Gliozzi and P. Provero, *The Svetitsky-Yaffe conjecture for the plaquette operator*, *Phys. Rev.* **D 56** (1997) 1131 [`hep-lat/9701014`].

[17] F. Gliozzi, *Screening of sources in higher representations of* SU($N$) *gauge theories at zero and finite temperature*, `PoS(LAT2005)196` [`hep-lat/0509085`].

[18] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333.

[19] E.P. Verlinde, *Fusion rules and modular transformations in 2-d conformal field theory*, *Nucl. Phys.* **B 300** (1988) 360.

[20] V.A. Fateev and A.B. Zamolodchikov, *Conformal quantum field theory models in two-dimensions having $Z_3$ symmetry*, *Nucl. Phys.* **B 280** (1987) 644.

[21] M. Caselle, G. Delfino, P. Grinza, O. Jahn and N. Magnoli, *Potts correlators and the static three-quark potential*, *J. Stat. Mech.* (2006) P03008 [`hep-th/0511168`].

[22] J. Fuchs, *Affine Lie algebras and quantum groups*, Cambridge University Press, 1992.

[23] A.N. Schellekens and S. Yankielowicz, *Extended chiral algebras and modular invariant partition functions*, *Nucl. Phys.* **B 327** (1989) 673.

[24] J. Engels et al., *A study of finite temperature gauge theory in (2+1) dimensions*, *Nucl. Phys.* **53** *(Proc. Suppl.)* (1997) 420 [`hep-lat/9608099`].

[25] M. Lüscher and P. Weisz, *Locality and exponential error reduction in numerical lattice gauge theory*, *JHEP* **09** (2001) 010 [`hep-lat/0108014`].

[26] J. Bagger, D. Nemeschansky and J.-B. Zuber, *Minimal model correlation functions on the torus*, *Phys. Lett.* **B 216** (1989) 320.

[27] J.L. Cardy, *Anisotropic corrections to correlation functions in finite-size systems*, *Nucl. Phys.* **B 290** (1987) 355.